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LOCATING A CRACK OF ARBITRARY BUT KNOWN SHAPE BY THE METHOD OF PATH-INDEPENDENT INTEGRALS

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Abstract—The solution of a crack problem of an arbitrary, but known, shape inside an infinite plane isotropic elastic medium can be achieved in general by the method of complex singular integral equations and their numerical solution by using the Gauss– or the Lobatto–Chebyshev methods. In a few special cases, like straight or circular-arc-shaped cracks, this solution is available in closed form. In this paper we will not contribute to the above methods of solution of crack problems, but we will propose a method for the determination of the exact position of such a crack inside a closed contour in the elastic medium by gathering and using information along this contour only (by experimental techniques) and applying the method of complex path-independent integrals for the location of the crack. This paper constitutes a nontrivial generalization of relevant previous results by the author and it is of quite general applicability in fracture mechanics for nondestructive testing. Numerical results for the particular case of a straight crack are displayed for the illustration of the efficiency of the method. The generalization of the present results to the determination of additional geometric and loading parameters of the crack is also suggested very briefly and related numerical results, concerning the length of the crack and the pressure distribution on it, in the aforementioned numerical application are also presented.

1. INTRODUCTION

We consider the problem described in some detail in the abstract. We notice that the solution of crack problems inside an infinite plane isotropic elastic medium can be found in closed form only in very special cases of the shape of the crack like straight or circular-arc-shaped cracks (Muskhelishvili, 1963). In the general case of curvilinear cracks, the corresponding problem can be solved numerically by a variety of methods. It is this author's and many other authors' opinion that the method of Cauchy-type singular integral equations (SIEs) is the most appropriate, especially when used in its complex form. Only one such equation, complex of course, has to be solved. The fundamental results on this method were presented by Ioakimidis (1976) together with a wide variety of applications and generalizations accompanied by numerical results. Some of the relevant results, in particular applications and generalizations for arbitrary curvilinear cracks or systems of such cracks, are those by Theocaris and Ioakimidis (1977a, 1979a, 1980a) and Ioakimidis and Theocaris (1977a, b, 1978, 1979a, b). It is not our intention to review here, even briefly, these references.

For the numerical solution of Cauchy-type singular integral equations, either in real or complex form, for crack problems of arbitrary shape we can use a variety of numerical techniques. It is generally accepted that the Gauss–Chebyshev method, originally due to Erdogan and Gupta (1972) and further improved by Theocaris and Ioakimidis (1979b), by using the natural extrapolation formula, as well as the Lobatto–Chebyshev method proposed by Ioakimidis (1976) and considered in further detail by Theocaris and Ioakimidis (1977b) are the most efficient ones. We will not review these methods here, but we notice that a long series of relevant papers is available in the literature. Under these circumstances the solution of a crack problem of arbitrary shape (or a system of such cracks) inside a finite or infinite plane isotropic or anisotropic elastic medium is very well established both from the theoretical and numerical points of view.

Of course, beyond complex Cauchy-type SIEs, one can also use a variety of additional numerical techniques for the solution of curvilinear crack problems including the classical

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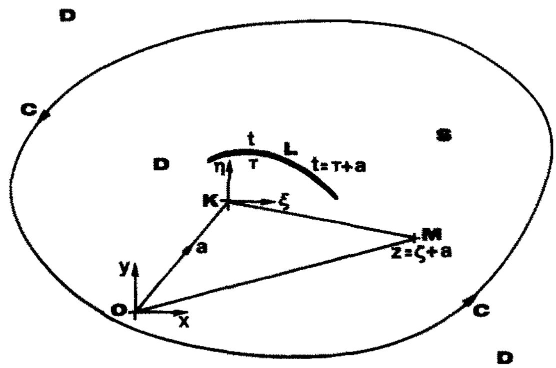

Fig. 1. Geometry of the crack L and the contour C in the general case of arbitrary curvilinear shapes of L and C.

and very popular boundary element method (BEM) [see, e.g. Brebbia *et al.* (1984) and Hartmann (1989) and the references therein], its complex-variable alternative (CVBEM), originally developed by Hromadka [see, e.g., Hromadka (1984), Hromadka and Pardoen (1985), Kassab and Hsieh (1990), and Hsieh and Kassab (1991) and the references therein] for two-dimensional potential problems, etc. These techniques can be applied to curvilinear crack problems in plane elasticity, exactly like the complex SIE method (as well as to many more problems) and are also particularly appropriate in this study. In fact, the SIE method is strongly related to BEM in spite of the different original approaches in these two methods and the fact that the SIE method is generally, but not always, used together with classical quadrature rules without "discretization" of the boundary as is the case in BEM. Because of our experience with the SIE method, we assume that this is the method used for the determination of the unknown function $g(t)$, proportional to the edge dislocation density along the crack L, below (on the understanding that BEM and CVBEM can also be successfully used), since we will consider only complex path-independent integrals and we will generalize complex-variable techniques to fracture mechanics. Finally, it is our personal opinion that the SIE method is more relevant to complex-variable techniques than the BEM and its variants. Of course, all of the above methods belong to the general category of boundary integral equations (BIEs).

In this paper we will use the method of complex path-independent integrals not for the solution of the above general crack problem, but for the location, that is, for the determination of the exact position, of the crack itself, if it exists, inside a sectionally smooth closed contour C surrounding this crack L (Fig. 1) on the basis of simple experimental data gathered by optical methods along C. We feel it necessary at first to briefly review the literature on complex path-indepedent integrals, since it is somewhat recent and not very well known. This will be done in the next section. Afterwards, in Section 3 we will proceed to the solution of our problem and we will show its effectiveness in Section 4 by a numerical application in the simple case of a straight crack determining also the crack length and the pressure distribution along the crack. Finally, in Section 5 we will discuss the proposed method and we will suggest possible generalizations of it.

2. COMPLEX PATH-INDEPENDENT INTEGRALS

It is very well known from the theory of analytic or, almost equivalently, holomorphic functions that if such a function $f(z)$ $(z = x+iy)$ is analytic in the domain S surrounded by a simple smooth closed contour C and on C as well, then the curvilinear complex integral of $f(z)$ on C vanishes, that is,

$$\oint_C f(z)\,dz = 0. \tag{1}$$

This is the classical Cauchy theorem in complex analysis, reported, e.g., by Copson (1976) and Churchill and Brown (1990). On the other hand, if $f(z)$ has one pole, $z = a$, inside C, then (1) takes the following modified form:

$$\oint_C f(z)\,\mathrm{d}z = 2\pi \mathrm{i} B, \tag{2}$$

where B is the residue of $f(z)$ at $z = a$. This is the classical Cauchy residue theorem in complex analysis proved also by Copson (1976) and Churchill and Brown (1990). The positive (anticlockwise) sense along C is assumed to be selected both in (2) and in the sequel (Fig. 1). The value of the integral in (2) does not depend on C, provided that $z = a$ lies in S (Fig. 1), and we can speak about a complex path-independent integral.

In plane elasticity problems, Budiansky and Rice (1973) expressed few classical real path-independent integrals appearing in fracture mechanics problems in complex form. One of these integrals was the classical and so popular J-integral. The classical complex potentials of Kolosov–Muskhelishvili $\phi(z)$ and $\psi(z)$, Muskhelishvili (1963), have been used for this purpose. Next, Theocaris and Ioakimidis (1980b) proposed the construction of an infinity of complex path-independent integrals in plane isotropic elasticity by using (2) and its generalizations with $f(z)$ combining $\phi(z)$ and/or $\psi(z)$ and/or other analytic functions. Of course, in that paper attention was paid to fracture mechanics and, especially, to the evaluation of stress-intensity factors. The case of inhomogeneous media with a crack along the interface was also considered by Ioakimidis (1980). The results of Theocaris and Ioakimidis (1980b) and Ioakimidis (1980) were generalized by Theocaris and Tsamasphyros (1980), Tsamasphyros and Theocaris (1982a, b) and Tsamasphyros (1981, 1989a, b) always with the applications to fracture mechanics taken into account. A new method for the construction of complex path-independent integrals for crack problems was also proposed by Ioakimidis (1987a) with applications to loaded straight cracks and to unloaded circular-arc-shaped cracks in plane isotropic elasticity as well as to unloaded straight cracks in plane anisotropic elasticity. More general and mathematically formulated results for complex path-independent integrals in plane isotropic elasticity were derived by Olver (1984, 1985) with reference to Tsamasphyros and Theocaris (1982a). Furthermore, complex path-independent integrals were also used for the solution of the problem of a straight crack in a finite plane isotropic elastic medium by Ioakimidis (1986a) and in plate problems by Ioakimidis (1992b). Several details on the practical application of the method are mentioned by Ioakimidis (1988a).

A probably more interesting application of complex path-independent integrals, like (2), is the location of zeros and/or poles of analytic and/or meromorphic functions. For the case of zeros we have the very old related results reported by Copson (1976) for the closed-form determination of inverse functions. In a more popular way, this method is described in brief by Householder (1970), whereas a simpler method was proposed by Abd-Elall *et al.* (1970). Another method, the Burniston–Siewert method, is also popular for sectionally analytic functions; it is described in detail by Henrici (1986). Additional results are mainly due to Ioakimidis and Anastasselou and were derived during the period 1984–87. A review of all of these results, including a very extensive list of references, was prepared by Ioakimidis (1987b).

In the case of poles of meromorphic (analytic with poles) functions, we can generalize the relevant methods for zeros of analytic functions. Related results were obtained by Abd-Elall *et al.* (1970) and Ioakimidis (1985a, 1986b) with a short review of these results by Ioakimidis (1987b) as well. As generalizations of these results we can report the location of essential singularities of a class of analytic functions, Ioakimidis (1988b), and, what is more important, the location of straight discontinuity intervals of arbitrary sectionally analytic functions, Anastasselou and Ioakimidis (1987), with an application to crack problems in fracture mechanics.

The idea beneath all of the above results is simply

- to use complex path-independent integrals for the location of cracks or other singularities in plane isotropic elasticity, which may cause the fracture of the specimen, as if these were poles of meromorphic functions.

Exactly as in the case of poles of meromorphic functions, we will study *an infinite medium D*, but with a Cartesian coordinate system Oxy arbitrarily defined in advance in an appropriate position of the infinite elastic medium. They are the distance and the polar angle of a point K rigidly bound to the geometry of the crack L with respect to this Cartesian coordinate system Oxy that will be determined. To this aim we will use an arbitrary sectionally smooth contour C (defining, e.g., a square, a circle, an ellipse, etc.), exactly as in (1) and (2), Fig. 1, completely surrounding the crack L and, probably, away from the crack that will be used. This contour is drawn in advance, without restrictions, by us. Obviously, the medium was aleady assumed infinite or, at least, very large compared to the dimensions of the crack L and the contour C and, therefore, the contour C *is not a boundary* of the elastic medium. Such a boundary does not exist or, if it does, it lies "at infinity". Of course, after having assumed a concrete contour C, the Cartesian coordinate system Oxy will be *rigidly attached* to this particular contour (Fig. 1). We can also add in passing that, as is clear from the references already reported, the case of zeros of analytic functions is essentially reduced to the case of poles of meromorphic functions, since the analytic function appears always in the denominator of the integrand. Thus a zero of this function, $h(z)$, is clearly a pole of $1/h(z)$.

It seems that the aforementioned generalization of the results about zeros and poles with complex path-independent integrals (both in the theory of analytic functions and in plane elasticity as well) to cracks is due exclusively to the author. In fact, Ioakimidis (1983) studied both cases of concentrated forces and of edge straight cracks, as well as short internal straight cracks, by the above method and the points of application of the concentrated forces, as well as the crack tips, were determined analytically on the basis of experimental data gathered by optical methods along a closed contour C only. A much more general case of location of a straight crack, under somewhat restrictive but sufficiently general loading conditions, in an infinite plane isotropic elastic medium was also considered later by Ioakimidis (1985b). Moreover, the location of straight-crack tips in finite plane isotropic elastic media was also studied by Ioakimidis (1986c). In some special cases, fracture can also be caused by holes and/or inclusions of a different material from that of the matrix. This case was considered by Ioakimidis (1987d) and for holes in a plate problem by Ioakimidis (1992b). Another important case is that of inclusions of arbitrary known shape, but of the same material with the matrix perfectly welded with the matrix, which was also studied by the above approach, Ioakimidis (1990). Finally, the orders of singularity can also be computed by using complex path-independent integrals (Ioakimidis, 1992a), as well as the location of branch points (Ioakimidis, 1992c).

In this way, the determination of zeros and/or poles of analytic and/or meromorphic functions (Ioakimidis, 1987b), has led, little by little, to a more or less powerful method for the location of cracks and other singularities in plane isotropic elasticity problems, Ioakimidis (1983, 1985b, 1986c, 1987d, 1990, 1992b, c), Anastasselou and Ioakimidis (1987), as if these singularities were poles of meromorphic functions. This seems to be an efficient method for nondestructive testing in finite domains S of the infinite elastic medium D where we have no accessibility of observation. Of course, we assume that we have accessibility of observation along the aforementioned closed contour C surrounding the crack L (Fig. 1) or some other geometrical or mechanical singularity, like a concentrated force or moment, and being the boundary of the finite region S under consideration. (Classical techniques of experimental stress analysis can be used to this purpose such as the pseudocaustics method.) The engineering and, further, industrial applications of this technique, probably automatic through a computer, are not clear and assured yet; nevertheless, they might be expected in due course. We can also add in passing the rather interesting generalization of the approach of path-independent integrals to the location of planar cracks in three-dimensional elasticity. Preliminary relevant results were proposed by Ioakimidis (1987c).

Of course, an alternative possibility is to use real path-independent integrals instead of complex path-independent integrals. Yet, in this case, the formulae will be much more complicated than the present ones based on the more or less "elegant" classical complex-variable theory.

For finite media one can also use the methods of singular and hypersingular integral equations (SIEs and HSIEs) appropriately modified for the detection and the determination of the shape of the crack L with the contour C coinciding with the boundary of the elastic medium D. This approach has been already successfully used, e.g. by Hartmann (1989), for the search of cavities in plane elastic media, Nishimura and Kobayashi (1990, 1991) in the potential case (harmonic, or Laplace, equation and related crack detection) and by Tanaka *et al.* (1990) in the elastodynamic case. This is surely a very interesting and useful possibility for finite regions D, easily generalizable to infinite regions on the basis of the present method, but it is also quite different from the present approach, which concerns infinite media D. Moreover, the present approach is a very simple one as a concept and as far as the formulae used are concerned contrary to the SIE and HSIE aforementioned approaches. On the other hand, as was already mentioned, the present results aim at a futher generalization of the classical and widely used complex-variable techniques for the location of zeros and poles to fracture mechanics, where the crack L is assumed to be a kind of "pole" of an appropriate complex potential. This might be an interesting possibility and, of course, completely impossible by the SIE and HSIE methods as these were used in the above and related references.

Furthermore, we can add that in practice we can measure very easily and sufficiently accurately the second derivative $\Phi'(z) \equiv \phi''(z)$ of the Kolosov–Muskhelishvili complex potential $\phi(z)$ (Muskhelishvili, 1963), with $\Phi(z) \equiv \phi'(z)$, inplane isotropic elasticity problems under generalized plane stress conditions at any point of the specimen by using elementary optical methods described in some detail by Theocaris and Ioakimidis (1980b) [see also Ioakimidis (1988a)]. This kind of measurement has been tested repeatedly in practice [e.g. by Theocaris and Razem (1977), Theocaris (1979), Theocaris *et al.* (1981)], with very good experimental results. We will not give further details on these methods here. We just mention that the term "pseudocaustic" for the image of the closed contour C (Fig. 1) on the screen seems rather completely unsuccessful in spite of the fact that the author has used this term himself too.

We ought also to add that in the method of the next two sections it will be the first derivative $\Phi'(z)$ of the classical complex potential $\Phi(z)$ of plane isotropic elasticity (Muskhelishvili, 1963) that will be used. The real part of $\Phi(z)$ is related to the sum of the principal stresses σ_{11} and σ_{22} at a point and, more explicitly, $\sigma_{11}+\sigma_{22} = 4\,\mathrm{Re}\,\Phi(z)$. On the other hand, $\mathrm{Im}\,\Phi(z)$ is the conjugate harmonic function of $\mathrm{Re}\,\Phi(z)$. Although $\Phi(z)$ can be used itself in the present results, we will use its first derivative $\Phi'(z)$ in the next two sections. Strangely enough, this derivative also has a simple physical meaning: as is very well known from the references in the previous paragraph $\Phi'(z)$ is simply proportional to the "complex" slope of the deformed surface of the specimen under generalized plane stress conditions as is assumed to be the case here. More explicitly, its real part is proportional to $\partial w/\partial x$ and its imaginary part to $-\partial w/\partial y$, where w denotes the third coordinate of the front or the rear originally (but not after loading) plane surface of the deformed specimen. As far as the proportionality constant is concerned, it is a very simple mechanical constant. Another mechanical–optical constant, quite easily determined and adjustable to our requirements is used in the pseudocaustics method. Therefore, $\Phi'(z)$ has a very simple physical meaning: combination of the two slopes just mentioned (with respect to x and y), and it does not refer to any "difficult" physical quantities or to quantities also having to do with the second complex potential $\Psi(z)$ (Muskhelishvili, 1963). Experiments by the pseudocaustics method, which is a very simple method, in the aforementioned and related references showed that it can lead to reliable results, the experimental error being in general satisfactorily small. We will not enter here into further details. Alternative experimental techniques are mentioned in Section 5.

Of course, as was also pointed out by Theocaris and Ioakimidis (1980b), the experimental approach of complex path-independent integrals with pseudocaustics along a closed contour C inside an infinite cracked medium D requires that small mirrors be put at the nodes to be used along this contour, on just one surface of the specimen, so that the values of $\Phi'(z)$ measured at these nodes may be directly used in the numerical integration rule as is explained in detail in the next section. Alternatively, a grid can be rigidly attached to the same surface (frequently the rear one) so that we can also know which point on the screen corresponds to a distinct point (node) on the contour C on the specimen. Evidently, in both cases, it is just the change of the "complex slope" after loading that is of interest.

We will proceed now to the main result of this paper, that is, the location of an arbitrary crack L, of known shape and loading conditions, in an infinite plane isotropic elastic medium by using the method of complex path-independent integrals, described in sufficient detail above and in more detail in the references of this section, together with the available experimental information for $\Phi'(z)$ along the known closed contour C surrounding the crack L (Fig. 1). Essentially, we will "see" the crack as *the pole of a meromorphic function and* **not** *as a curve of discontinuity* L *of* $\Phi'(z)$ as is really the case. An analogous approach for inclusions of the same material with the matrix and welded with the matrix was already followed by Ioakimidis (1990).

3. LOCATION OF THE CRACK

We consider the crack L of Fig. 1 of arbitrary, but known, shape and under arbitrary, but also known, loading conditions inside the sectionally smooth closed contour C (Fig. 1). We use simultaneously two Cartesian coordinate systems with parallel axes: Oxy (rigidly attached to the contour C as was already mentioned) with its position known to us and $K\xi\eta$ (rigidly attached to the crack L) with its centre K not known to us. We denote by $z = x+\mathrm{i}y$ and $\zeta = \xi+\mathrm{i}\eta$ the corresponding complex variables. It is sufficient to determine the complex coordinate $a = a_1+\mathrm{i}a_2$ of the point K in the Oxy Cartesian coordinate system. Then the exact location of the crack L will have been determined. Clearly, under these circumstances, we have (in complex notation): $z = \zeta+a$ or $\zeta = z-a$. For the points of the crack L itself, we use the symbols t and τ instead of z and ζ, respectively (Fig. 1).

Since the plane isotropic elastic medium was assumed infinite or, clearly, at least of sufficiently large dimensions compared to those of C and L in Fig. 1, the complex potential $\Phi(z) \equiv \phi'(z)$ (Muskhelishvili, 1963), will be of the form (Ioakimidis, 1976)

$$\Phi(z) = \Gamma + \frac{1}{2\pi \mathrm{i}} \int_L \frac{g(t)}{t-z}\,\mathrm{d}t, \tag{3}$$

where Γ is a constant of no interest here because of (1). For the crack L in an infinite isotropic elastic medium it is possible to determine the density $g(t)$ in the Cauchy-type integral (3) by using the method of complex Cauchy-type singular integral equations (Ioakimidis, 1976), with sufficient relevant references given in Section 1, or the additional methods reported in the same section. The crucial point in this way of thinking is that $g(t)$ does not depend on the position of L, since we have an infinite medium or, at least, of very large dimensions compared to those of C and L. Therefore, we can assume without doubts that $g(t)$ in (3) is a known function, having been determined in practice by the methods reported in Section 1. Moreover, since $g(t)$ does not depend on the position of L inside C, we can write $g(t) \equiv g^*(\tau)$ with $g^*(\tau)$ also being a known function. More explicitly, for every point τ of L we have a particular value of $g^*(\tau)$ corresponding to this point and this value is available to us (after the required computational effort, of no interest in this paper).

Now, by taking into account that $t = \tau+a$ or $\tau = t-a$ for the points of the crack L (Fig. 1), we can rewrite (3) as

$$\Phi(z) = \Gamma + \frac{1}{2\pi \mathrm{i}} \int_L \frac{g^*(\tau)}{\tau-(z-a)}\,\mathrm{d}\tau. \tag{4}$$

Furthermore, since it is $\Phi'(z)$ which is available experimentally along C, we differentiate once (4) and we obtain

$$\Phi'(z) = \frac{1}{2\pi i}\int_L \frac{g^*(\tau)}{[\tau-(z-a)]^2}\,d\tau. \tag{5}$$

At this point, we use the classical geometric series (Churchill and Brown, 1990), appropriately modified in our case, and we assume that $|\tau| < |\zeta| = |z-a|$ in the denominator of (5), which is really the case if C lies somewhat away from the crack L as in Fig. 1 and/or the point K, rigidly attached to the crack L, lies close to the crack. Next, by differentiating the elementary geometric series, applied here to the denominator of (5), we find directly for $\Phi'(z)$ that

$$\Phi'(z) = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{A_k}{(z-a)^{k+2}}. \tag{6}$$

In this equation we have introduced the new quantities

$$A_k = (k+1)\int_L \tau^k g^*(\tau)\,d\tau, \quad k = 0, 1, \ldots, \tag{7}$$

which are related only to the crack L and not to the contour C. Furthermore, since $g^*(\tau)$ is a known function, of course after the numerical solution of the corresponding crack problem in our case, A_k are also known quantities in advance.

Now we can proceed to the definition of the fundamental complex path-independent integrals I_m along the closed contour C (always in the anticlockwise sense)

$$I_m = \oint_C z^m \Phi'(z)\,dz, \quad m = 0, 1, \ldots. \tag{8}$$

Since $\Phi'(z)$ is available along C through experimental methods, the values of I_m are also available to us, usually by using the trapezoidal or some other, frequently equispaced and rather simple, quadrature rule (Davis and Rabinowitz, 1984). Beyond numerical analysis books, the literature on boundary element techniques reports several such more or less elementary and efficient rules. The increase in the number of nodes n used generally gives much better numerical results for I_m exactly as the increase in the accuracy of the experimental data for $\Phi'(z)$ along C by paying more attention to the execution of the relevant experiment for the determination of $\Phi'(z)$ along C, usually at the nodes of the quadrature rule. Next, by using the classical Cauchy residue theorem (2) (Churchill and Brown, 1990), we have

$$\oint_C (z-a)^m \Phi'(z)\,dz = A_{m-1}, \quad m = 1, 2, \ldots, \tag{9}$$

because of the expression (6) for $\Phi'(z)$. In the special case where $m = 0$ (9) takes the following very simple form

$$I_0 \equiv A_{-1} = \oint_C \Phi'(z)\,dz = 0, \tag{10}$$

where (6) and (8) were also taken into account. This equation is also clear from (5) and can be used in practice as a simple method of checking the accuracy of the experimental data as well as the accuracy of the numerical integration rule used. We will not use (10) in the sequel.

Returning to (9) and taking into account (8) and (10), we find directly the linear expressions giving A_{m-1} in terms of I_m ($m = 1, 2, \ldots$). These expressions can easily be derived by using the classical table of binomial coefficients (Dwight, 1961). For the sake of space we will not display these expressions here. Nevertheless, we will give the expressions for I_m in terms of A_{m-1} (again with $m = 1, 2, \ldots$)

$$A_0 = I_1 \quad \text{for} \quad m = 1, \tag{11a}$$

$$A_1 + 2aA_0 = I_2 \quad \text{for} \quad m = 2, \tag{11b}$$

$$A_2 + 3aA_1 + 3a^2A_0 = I_3 \quad \text{for} \quad m = 3, \tag{11c}$$

$$A_3 + 4aA_2 + 6a^2A_1 + 4a^3A_0 = I_4 \quad \text{for} \quad m = 4, \tag{11d}$$

$$A_4 + 5aA_3 + 10a^2A_2 + 10a^3A_1 + 5a^4A_0 = I_5 \quad \text{for} \quad m = 5, \tag{11e}$$

$$A_5 + 6aA_4 + 15a^2A_3 + 20a^3A_2 + 15a^4A_1 + 6a^5A_0 = I_6 \quad \text{for} \quad m = 6, \tag{11f}$$

and so on. These equations can be derived either from the solution of the aforementioned linear expressions of A_{m-1} in terms of I_m or, alternatively, simply by taking into account that $z^m = [(z-a)+a]^m$.

At this point, we can add in passing that, quite frequently, the resultant force of the loading on the edges of the crack L vanishes, e.g. in the cases of unloaded cracks. In this special case, we have from (7)

$$A_0 = I_1 = \int_L g^*(\tau)\,\mathrm{d}\tau = 0 \tag{12}$$

because of (11a) too (Muskhelishvili, 1963). In this special, but quite frequent in practice case, (11a) becomes useless, whereas (12) can be used, exactly as (10), for the verification of the accuracy of the experimental data and of the efficiency of the numerical integration rule.

Now we observe from (11) that we have an infinite system of equations for the determination of only one unknown complex quantity, a, the position of K in the complex plane (Fig. 1), which is completely sufficient for the location of the crack L. Frequently, we use (11b) for the determination of a, that is,

$$a = (I_2 - A_1)/(2I_1) = (I_2 - A_1)/(2A_0) \tag{13}$$

because of (11a). Next, we can check the resulting value of a by using a few of the subsequent equations (11). Of course, in the case where L is unloaded (loading only at infinity or far from the crack L), A_0 and I_1 vanish because of (12). Therefore, (13) cannot be used in this very important case in engineering applications. In this case, we can simply use (11c), which yields

$$a = (I_3 - A_2)/(3I_2) = (I_3 - A_2)/(3A_1) \tag{14}$$

because of the fact that $A_1 = I_2$ in this particular case as is clear from (11b) since (12) holds true.

We conclude this section with two remarks:

(i) Since the experimental data for $\Phi'(z)$ along the closed contour C are available, we can compute at first the integrals I_m from (8) and use, afterwards, (11) for the determination of a. Clearly, beyond this determination, essentially the location of the crack L as was already mentioned, the use of additional equations from the set of (11) permits us to receive valuable further information on the geometry and/or loading of the crack L, probably not available in advance. For example, in the application of the next section, we will determine not only the position of the crack L, but also its size (length in this application) $2b$ as well as the loading intensity p. A further related possibility concerns the determination of the angle of orientation α of the real crack L in Fig. 1 with respect to the assumed orientation of the same crack in the preliminary theoretical/numerical determination of $g^*(\tau)$ in the "generic" solution as was already explained in detail. This can be easily achieved by using higher-order complex path-independent integrals I_m in the system of equations (11), but, in practice, it is clearly understood that higher experimental accuracy will be required as well, since the numerical errors in I_m will be combined for an increasing number of unknowns (either geometrical: position, size, orientation, slope, curvature and so on, or mechanical: loading distribution determined by one or more parameters or simply loading at infinity) for the crack L.

(ii) Evidently, if no crack exists in the elastic medium inside the closed contour C, $\Phi(z)$ is an analytic function and not a sectionally analytic function as was assumed in the present section in the finite domain S of the infinite elastic medium D bounded by C (Fig. 1). It is obvious that in this case the integrals I_m, defined by (8), will vanish identically. This remark may be of some interest during the potential application of the theoretical results of this paper to nondestructive testing in future when one has to decide about the existence or inexistence of a crack L in an inaccesible region S of a plane isotropic elastic medium D and, in the second case, proceeding to its location and to the determination of a few of its geometric properties and/or loading conditions. Of course, these future possibilities exceed the author's aim in this paper, where just the fundamental idea for the location of a crack L is studied.

4. AN APPLICATION

In this section we will apply the results of the previous section to a very simple, yet nontrivial, case. We assume that the region S is a circle of radius R bounded by its circumference C. This boundary C of S was selected, for the sake of convenience, by us although the real elastic medium D remains infinite. The centre O of the Cartesian coordinate system Oxy is assumed to coincide with the centre of this circle S (Fig. 2). In order to reduce the computational effort, that is, to avoid calculations with complex numbers, we have restricted our attention to a crack L along the real axis Ox with midpoint at the point $x = a$ and length $2b$. These assumptions do not reduce the generality of the method of the previous section, aiming simply at its illustration in a rather simple mechanical/geometrical environment.

It is very well known that, generally, $g^*(\tau)$ presents inverse-square-root power singularities at the crack tips (Muskhelishvili, 1963). In our case, we select the following simple form of $g^*(\tau)$:

$$g^*(\tau) = -2pi\tau/\sqrt{b^2-\tau^2}, \tag{15}$$

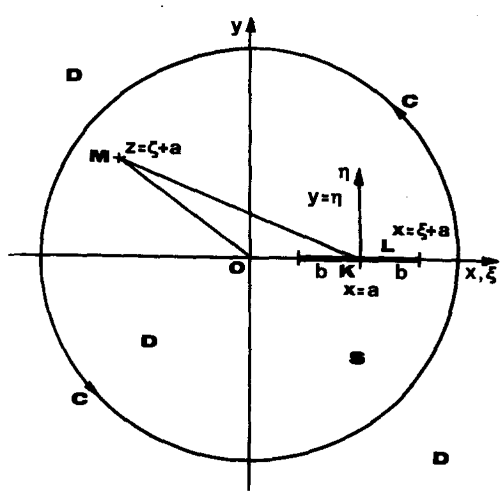

Fig. 2. Geometry of the straight crack L and the circular contour C in the case of the application.

where p denotes a constant pressure with pressure distribution units exactly as $\Phi(z)$ itself. Then we obtain from (4) for $\Phi(z)$

$$\Phi(z) = \Gamma - \frac{p}{\pi} \int_{-b}^{b} \frac{1}{\sqrt{b^2-\tau^2}} \frac{\tau}{\tau-(z-a)} \mathrm{d}\tau. \tag{16}$$

At this point, we can add in passing that just in the case of a straight crack, if we have a tensile loading only at infinity and parallel to the crack, then our fundamental complex function $\Phi'(z)$ vanishes, exactly as is also the case with the stress-intensity factors at the crack tips, and the method fails. This is an exceptional "pathological" case and we have to change the orientation of the loading.

On the other hand, we have available the following classical elementary formula (Chawla and Ramakrishnan, 1974),

$$\frac{1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-\tau^2}} \frac{T_n(\tau)}{\tau-\zeta} \mathrm{d}\tau = -\frac{1}{\sqrt{\zeta^2-1}\,[\zeta+\sqrt{\zeta^2-1}]^n}, \quad \zeta \notin [-1, 1], \tag{17}$$

where $T_n(\tau)$ denotes the classical Chebyshev polynomial of the first kind and degree n. For $n = 1$, $T_n(\tau) = \tau$ and (17) reduces to the even more elementary formula

$$\frac{1}{\pi} \int_{-b}^{b} \frac{1}{\sqrt{b^2-\tau^2}} \frac{\tau}{\tau-\zeta} \mathrm{d}\tau = 1 - \frac{\zeta}{\sqrt{\zeta^2-b^2}} = 1 - \frac{1}{\sqrt{1-(b/\zeta)^2}}, \tag{18}$$

written here for the interval $[-b, b]$ instead of $[-1, 1]$. This is clearly the case of a crack under constant pressure distribution or uniform tensile loading at infinity of direction normal to that of the crack, that is along the Oy-axis in Fig. 2 (Muskhelishvili, 1963). These cases coincide in this paper since just $\Phi'(z)$ is used. In fact, because of (18), (16) yields, since $z-a = \zeta$; $z = x+\mathrm{i}y$, $\zeta = \xi+\mathrm{i}\eta$,

$$\Phi(z) = \Gamma - p + \frac{p\zeta}{\sqrt{\zeta^2-b^2}}, \quad \zeta = z-a, \quad \zeta \notin [-b, b], \tag{19}$$

whence

$$\Phi'(z) = -\frac{pb^2}{\sqrt{\zeta^2-b^2}^{\,3}} = -\frac{pb^2}{\sqrt{(z-a)^2-b^2}^{\,3}}, \quad \zeta \notin [-b, b]. \tag{20}$$

In our case, we assume that all three constants: a for the location of the crack, b for the half-length of the crack and p for the pressure distribution along the crack edges are not known in advance. In more complicated applications, it would be possible to have more than three unknown quantities, e.g. the orientation α of the crack with respect to the far-field loading (or, almost equivalently, the Ox-axis) and the additional quantities already mentioned in the previous section, although a is of primary importance in the present paper. Of course, the requirement for the determination of more than one fundamental quantity, including, e.g. the crack position a, the orientation of the crack α, the crack length $2b$, the loading intensity p, etc., will lead to the necessity of using higher-order complex path-independent integrals and much more computational effort. Therefore, beyond increased numerical accuracy in the quadrature rules used, higher experimental accuracy will also be necessary.

On the other hand, it should be observed again that in a real engineering environment $\Phi'(z)$ would be obtained from more or less accurate measurements along the circumference C of the circle S of Fig. 2. No closed-form formula, like (20), would be available. Nevertheless, for convenience and without the slightest loss of generality, we will use the numerical values for $\Phi'(z)$ obtained from (20) along C only as if they were obtained from the experiment itself. We can proceed now to the required numerical results.

At first, we use (7) for the evaluation of the quantities A_k. In our case, because of (15) and putting $\tau = bv$, we find easily that

$$A_k = -2pi(k+1)b^{k+1}\int_{-1}^{1}\frac{v^{k+1}}{\sqrt{1-v^2}}\,dv. \tag{21}$$

For even values of $k: k = 2l$ ($l = 0, 1, \ldots$), it is directly clear from (21) that A_k vanishes, whereas this is not true in the case of odd values of $k: k = 2l-1$ ($l = 1, 2, \ldots$). In this case, (21), on the basis of elementary formulae reported by Gradshteyn and Ryzhik (1980) reduces to

$$A_{2l-1} = -2pi\pi b^{2l}\frac{1\cdot 3\cdot\ldots\cdot(2l-1)}{2\cdot 4\cdot\ldots\cdot 2(l-1)},\quad l = 1, 2, \ldots. \tag{22}$$

More explicitly, we find from (22)

$$A_1 = -2pi\pi b^2,\quad A_3 = -3pi\pi b^4,\quad A_5 = -3.75pi\pi b^6, \tag{23}$$

and so on.

On the other hand, for the integrals I_m, defined by (8), we have used for their evaluation (20) together with the classical trapezoidal quadrature rule with n nodes along the circumference C of the circle S (Fig. 2). The classical trapezoidal quadrature rule has the following form for periodic functions with period equal to 2π

$$\int_0^{2\pi} h(\theta)\,d\theta \simeq \frac{2\pi}{n}\sum_{k=1}^{n} h(\theta_k),\quad \theta_k = \frac{(2k-1)\pi}{n},\quad k = 1, 2, \ldots, n, \tag{24}$$

the error term E_n being ignored (Davis and Rabinowitz, 1984). Then we obtain

$$\oint_C f(z)\,dz = iR\int_0^{2\pi} f(Re^{i\theta})\,e^{i\theta}\,d\theta \tag{25}$$

for an analytic function $f(z)$ along C only as is really the case in (8), where $f(z) = z^m\Phi'(z)$ with $\Phi'(z)$ given by (20) in the present application. We can now apply (24), with the nodes given in this formula or any other set of nodes differing from these nodes by a constant, e.g. $\theta_k = 2k\pi/n$, $k = 0, 1, \ldots, n-1$ or $k = 1, 2, \ldots, n$, to the evaluation of the complex contour integral (25). We do not feel it necessary to give additional relevant details. We restrict ourselves to the remark that, as is clear in the present special application, the integrals I_m evaluated along C (Fig. 2) are purely imaginary quantities, because of (8), (20) and (25) for real values of the geometric parameters a and b.

Moreover, it is obvious from (8) that the first two integrals I_0 and I_1 vanish as can easily be seen by using the path-independence of I_m as the radius R of C tends to infinity. Finally, we notice that in practice (in the present special application), because of the fact that I_m/i take real values for all values of m ($m = 0, 1, 2, \ldots$) it is sufficient to use only the $n/2$ nodes $z_k = Re^{i\theta_k}$ in the computer, lying in the upper half of C, provided that the nodes in (24) are selected, but this is of marginal importance.

In Table 1 we present the computed numerical values of the quantities

$$I_m^* = I_m/(-\pi i) = iI_m/\pi,\quad m = 0, 1, \ldots, 6, \tag{26}$$

by the above-described trapezoidal quadrature rule for $R = 10$, $a = 2$, $b = 3$ and $p = 1$, as well as for $n = 4$, 8, 16 and 32 nodes. These results are displayed for $m = 0, 1, \ldots, 6$, that is, they concern the first seven of the complex path-independent integrals I_m as these were slightly modified by using (26). In the same table we also display the values of the quantities

Table 1. Numerical results for the integrals I_m^* ($m = 0, 1, \ldots, 6$), computed by using (8), (20) and (26) together with the trapezoidal quadrature rule with $n = 2^j$ ($j = 2, 3, 4, 5$) nodes, the quantities A_m^* ($m = 3, 4, 5$), computed by using (28) [and *not* (7)], the position a of the centre of the crack, computed by using (30), the half of the length of the crack b, computed by using (31), and the tensile loading p, computed by using (32), in the case where $R = 10$, $a = 2$, $b = 3$ and $p = 1$

	$n = 4$	$n = 8$	$n = 16$	$n = 32$
I_0^*	−0.06157348	−0.00635665	−0.00003653	+0.00000000
I_1^*	−0.35527314	−0.03397704	−0.00018864	+0.00000000
$I_2^* = A_1^*$	+16.005091	+17.819807	+17.999028	+18.000000
I_3^*	+097.04795	+107.05027	+107.99500	+108.00000
I_4^*	+615.73484	+670.01942	+674.97431	+675.00000
I_5^*	+3552.7314	+3843.9889	+3869.8682	+3870.0000
I_6^*	−160050.91	+21498.379	+21633.075	+21633.750
A_3^*	+223.43032	+241.29196	+242.99099	+243.00000
A_4^*	−26.779439	−02.745123	−00.015862	+00.000001
A_5^*	−177424.14	+2720.4066	+2733.6817	+2733.7500
a	+2.0211892	+2.0024585	+2.0000154	+2.0000000
b	+3.0506779	+3.0045144	+3.0000254	+3.0000000
p	+0.8598753	+0.9870165	+0.9999291	+1.0000000

$$A_m^* = A_m/(-\pi \mathrm{i}) = \mathrm{i}A_m/\pi, \quad m = 3, 4, 5, \tag{27}$$

determined by using the equations

$$A_1^* = I_2^* \quad \text{for} \quad m = 1, \tag{28a}$$

$$A_2^* = I_3^* - 3aI_2^* \quad \text{for} \quad m = 2, \tag{28b}$$

$$A_3^* = I_4^* - 4aI_3^* + 6a^2I_2^* \quad \text{for} \quad m = 3, \tag{28c}$$

$$A_4^* = I_5^* - 5aI_4^* + 10a^2I_3^* - 10a^3I_2^* \quad \text{for} \quad m = 4, \tag{28d}$$

$$A_5^* = I_6^* - 6aI_5^* + 15a^2I_4^* - 20a^3I_3^* + 15a^4I_2^* \quad \text{for} \quad m = 5, \tag{28e}$$

strongly related to (11) and with the previous remarks taken into account. On the other hand, the exact values of the above quantities are

$$A_1^* = 2pb^2, \quad A_2^* = 0, \quad A_3^* = 3pb^4, \quad A_4^* = 0, \quad A_5^* = 3.75pb^6, \tag{29}$$

where (23) were also taken into consideration.

Moreover, by using (28b) and the second of (29), we have for the approximate value of a

$$a = I_3^*/(3I_2^*). \tag{30}$$

This is the value used in (28c–e) for the approximate evaluation of A_3^*, A_4^* and A_5^*, displayed in Table 1, since the true value of a, $a = 2$ in our case, is not known in advance. On the other hand, we observe from the first and the third of (28) that

$$b = \sqrt{2A_3^*/(3A_1^*)}. \tag{31}$$

This is the approximate value of b, displayed in Table 1 for $n = 4$, 8, 16 and 32 nodes, since its true value, $b = 3$ in our case, is not known in advance.

Finally, from the same equations we obtain for p

$$p = 3A_1^{*2}/(4A_3^*). \tag{32}$$

This is the approximate value of p, displayed in Table 1 for $n = 4$, 8, 16 and 32 nodes, since its true value, $p = 1$ in our case, is similarly not known in advance. The formulae (30), (31) and (32) permit us, after the evaluation of I_m^* by using only experimental data and A_m^* by using (28), the approximate evaluation of the three quantities of interest in our case: a, b and p, the first two of which concern the geometry of the crack, whereas the third one its loading, tensile in our case.

In a similar way, we present in Table 2 completely analogous numerical results for the case when $R = 20$, $a = -10$, $b = 5$ and $p = 1$ by using now $n = 16$, 32, 64 and 128 nodes in the trapezoidal quadrature rule (24).

From the numerical results of Tables 1 and 2 we observe the extremely rapid convergence of these results as the number of nodes n in the trapezoidal quadrature rule increases. Moreover, we observe the accuracy of the obtained numerical results, especially for a, b and p, which is at least about eight significant digits for $n = 32$ in Table 1 and $n = 128$ in Table 2. The somewhat lower accuracy in the numerical results of Table 2, compared to those of Table 1, is simply due to the fact that the crack L is very long, $2b = 10$, in comparison with the radius of the circumference of the circle C, $R = 20$, where the numerical data were gathered. Moreover, the large value of $|a|$, $|a| = 10$, is an additional relevant reason. This is not the ordinary case in practice in the light of the aims of this paper: nondestructive testing by using data gathered far away from the singularity which may cause fracture, the crack L in our case. This behaviour of the numerical results, that is more accurate numerical results for larger values of R and smaller values of a and b, was really observed in additional computations, beyond those of Tables 1 and 2, as well. Of course, in experiments we cannot ignore the opposite fact that more accurate numerical results for $\Phi'(z)$ are obtained near the crack L than far away from L. More accurate experimental techniques are required in this case.

Finally, we observe from the numerical results of Tables 1 and 2: (i) the obvious fact that $A_1^* = I_2^*$, due to (28a) in our application; (ii) the similarly obvious fact that $I_0^* = I_1^* = A_4^* = 0$ as $n \to \infty$. As was already mentioned, this last fact confirms the correctness of our computations and it is useful as a check of these computations. We will not proceed to further discussion on the numerical results of Tables 1 and 2, which are sufficiently clearly displayed, generally in eight significant digits.

5. CONCLUSIONS—DISCUSSION

We have used the above method of complex path-independent integrals for the location of a crack (of arbitrary shape) inside an infinite plane isotropic elastic medium as if this crack were a pole of a meromorphic (analytic with poles) function. We have seen, especially in the previous section, that we may have unknown not only the position of the crack, but also other quantities concerning the geometry and/or the loading conditions of the crack. We have also observed the efficiency of the method in the numerical results of the previous

Table 2. Analogous results to those of Table 1, but for $j = 4, 5, 6, 7$, $R = 20$, $a = -10$, $b = 5$ and $p = 1$

	$n = 16$	$n = 32$	$n = 64$	$n = 128$
I_0^*	−0.01701279	−0.00025338	−0.00000004	+0.00000000
I_1^*	+0.26424358	+0.00386359	+0.00000055	+0.00000000
$I_2^* = A_1^*$	+45.905079	+49.941118	+49.999992	+50.000000
I_3^*	−1436.6685	−1499.1031	−1499.9999	−1500.0000
I_4^*	+30897.249	+31861.344	+31874.998	+31875.000
I_5^*	−578678.60	−593542.16	−593749.97	−593750.00
I_6^*	+10139105.0	+10367932.0	+10371093.0	+10371094.0
A_3^*	+0922.1163	+1861.8816	+1874.9981	+1875.0000
A_4^*	−9404.2111	−0112.8410	−0000.0158	+0000.0000
A_5^*	−110337.12	+56519.084	+58593.462	+58593.747
a	−10.432168	−10.005804	−10.000001	−10.000000
b	+3.6594586	+4.9854145	+4.9999979	+5.0000000
p	+1.7139457	+1.0046753	+1.0000007	+1.0000000

section (Tables 1 and 2) and we essentially expressed the opinion that the results of the present paper, although theoretical in principle, might become in future the background for the development of a viable engineering/industrial technique of nondestructive testing about the existence or inexistence of a crack in a region S where we have no accessibility of observation beyond its boundary C (Fig. 1 or Fig. 2 for the application of the previous section). Unfortunately, we do not have available the experimental and relevant facilities for the development of this technique ourselves.

We have already used two "vehicles" in our results: (i) the first derivative $\Phi'(z)$ of the classical complex potential $\Phi(z)$ of Kolosov–Muskhelishvili, and (ii) the experimental method of pseudocaustics. This happened because of the simplicity of these "vehicles" and our experience with them in previous results by us. In Section 2 we had also the opportunity to explain in some detail the simple physical meaning of $\Phi'(z)$ (as the pair of slopes of the specimen surface in complex form). Similarly, the experimental method of pseudocaustics, based just on the reflection and, usually, diffraction as well of light rays, is also a very simple experimental method. Nevertheless, experimental stress analysis offers a variety of well-known techniques, some of which are much more accurate than the method of pseudocaustics, which can also be used. These techniques include interferometry, holography, photoelasticity, moiré fringes, speckle patterns, diffraction methods, optical sensors, their various combinations, etc. We are not experts in these methods, we will not analyse the corresponding experimental errors and we simply make reference to the related literature, e.g. in the journals *Experimental Mechanics* and *Experimental Techniques*. What is sure is that one can use experimental data along the closed contour C and, probably, in its close neighbourhood in order to be able to derive any required results for the stress, strain and displacement components in the specimen and for their derivatives. Yet, quite frequently, the experimental data should be combined with appropriate related numerical techniques. This means that in practice we have not to restrict ourselves to $\Phi'(z)$ as a "vehicle" for the derivation of our method, but we can also use any other complex potential or combination of complex potentials. $\Phi(z)$ is such a good candidate and the required modification of the formulae of the previous two sections is rather trivial and will be omitted.

On the other hand, it is clear that any computational technique appropriate for plane elasticity problems, like the SIE and the HSIE methods, can be used in the present problem, supplied with the appropriate set of data, so that the location, the orientation and additional parameters of a crack can be derived. One has just to solve the necessary number of linear equations accompanied by one or more nonlinear equations, and a powerful computer environment is sufficient. This is the "direct" approach to the problem. Here the approach is different: it aims at as simple results as possible and it is based on the generalization of the classical complex-variable techniques for zeros and poles of analytic/meromorphic functions and on complex path-independent integrals. It "sees" the crack as a "pole" of a meromorphic function in complex analysis. In this sense, the present results are quite different from those by the SIE and HSIE related methods, some of which were referenced in Section 2. Finally, the present method concerns infinite media (and not finite media as the previous methods) and it uses an arbitrary closed contour C surrounding the crack L and not the external boundary Γ of the medium. Of course, the present method can, in principle, easily be generalized for finite media by using complex Cauchy-type SIEs or HSIEs, where both the external boundary Γ and the crack L will be simultaneously taken into account. The related numerical solution and the determination of the unknown parameters of the crack (on the basis of the available data) can be made either by direct or by indirect (iterative) methods such as the well-known and widely used "alternating method". In such a case of a finite medium, the contour C may coincide with the external boundary Γ of the elastic medium D or it may not, but in the latter case much more computational effort will be required. In any case, this author is convinced that complex-variable techniques, like complex path-independent integrals in this paper, lead to simpler equations than their real-variable alternatives in classical plane elasticity.

Another question concerns which equations out of (11) should be used. In the application in Section 4 we used the first nontrivial one(s), that is that/those corresponding to the smaller integer(s) m. This is natural. Moreover, this is also advisable, since the computation of the integrals in (8) becomes more "difficult" from the numerical analysis point of view for higher values of m and, therefore, in general, more nodes are required in this case. Of course, it is clearly understood that, because of unavoidable errors in experimental data and in numerical integration, the numerical results for the crack location and, possibly, the additional required parameters, will be slightly different if another equation or a set of equations (11) is used. Normally, the difference will not be significant. If it is, higher accuracy in the experimental data and in numerical integration will be required. Alternatively, if we wish, we can use a larger number of equations (11) than that really required for the determination of the unknown parameters and proceed by a least-square approach. In principle, this technique cannot be recommended because of the relative increase in the errors of numerical integration for higher values of m in (11). Alternatively, the most reliable value of a is generally that derived from that equation out of (11) which corresponds to the lowest value of m. In any case, from the theoretical point of view, a in (11) satisfies all of these equations simultaneously; it is only the experimental and numerical errors that may lead to slightly different numerical results depending on the equation chosen for the determination of a. By no means can we speak in the present case about nonuniqueness in the solution of our problem, but we can surely speak about the influence of errors (experimental and numerical) on the final results. These are, of course, unavoidable in most computations.

A quite similar problem arises in singular integral equations (SIEs and HSIEs). When solving such an equation by the quadrature method, we approximate it by a numerical integration rule. Next, the approximate equation is applied at a set of nodes, which can generally be selected arbitrarily. If we use the whole interval (a, b) of integration as nodes, we obtain an infinite number of equations for our infinite collocation points. But, generally, we are satisfied by using a simple set of nodes and the related equations exactly as we have selected in Section 4 the simplest equations out of (11) and we understand that another selection of the nodes will lead to slightly different results, where, of course, the accuracy of the numerical integration rule is of fundamental importance. Exactly similar is the collocation method for SIEs and HSIEs although, obviously, one can attempt the optimization of the numerical results (solution of the SIE or HSIE, determination of the stress-intensity factors, etc.) by the least-square method or the similar Galerkin method with quite a doubtful improvement of the numerical results for the *same* computational effort. This case is completely analogous to our case with (11). Of course, exactly as nobody can claim about nonuniqueness in a SIE or HSIE because of the slight difference of the results when collocation points are chosen in a different way, similarly, we cannot speak about nonuniquenes of our numerical results because of the selection of the simplest equations out of (11). Nonuniqueness concerns important problems in a mathematical formulation where this formulation does not lead, theoretically, to just one solution and has nothing to do with the influence of numerical and experimental errors on the results.

In Section 2 we have had the opportunity to make reference to several results for the location of cracks, holes and inclusions and the determination of their parameters in an elastic medium. This is also the problem having been studied here. This problem is not the classical problem of elasticity, where the position of the crack is known in advance and just its solution is required. In this sense, we can say that here (as well as in the aforementioned references) we have an inverse crack problem and we have to determine the crack location and, possibly, additional parameters (as in Section 4). Although the present problem can clearly be labeled as an inverse one, by no means is it an ill-posed problem (as is the case, e.g., in some integral equations of the first kind, but not in SIEs or HSIEs). The numerical results of Section 4 verify the well-posedness of our problem, but, of course, it is also well understood that the present technique is just a generalization of the related complex-variable techniques for zeros/poles of analytic/meromorphic functions already reported in brief in Section 2. All of these methods are very well posed and completely assure the uniqueness of their results. Quite similar is their present generalization to fracture mechanics problems.

Finally, just for the better understanding of the principle of the method, in which we have been particularly interested, and for the sake of convenience as well, we have restricted our attention to the case of one crack and to the case of a homogeneous infinite plane isotropic elastic medium. Yet, clearly, the previous results can be easily generalized to the case of a system of cracks [instead of only one crack (Figs 1 and 2)], as well as to the cases of a combination of cracks with holes and/or inclusions always for an infinite plane isotropic elastic medium. In the case of anisotropic elastic media, the method becomes sufficiently complicated with the relevant experimental techniques most probably inapplicable. Similarly, the present technique is only approximately valid for media D of sufficiently large dimensions compared to those of the crack L and the contour C as well (Figs 1 and 2) although it is generalizable to finite media as was mentioned previously. Moreover, of particular importance may be the cases of arrays of periodic, doubly-periodic and star-shaped cracks or the case of a half-plane or even two welded along their interface half-planes of different materials. We can easily generalize our method to these cases, taking also into account the relevant references reported in Section 1. Additional easy generalizations of the present method seem also quite possible and expected by the author to appear in the future by himself and/or by other interested researchers.

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REFERENCES

Abd-Elall, L. F., Delves, L. M. and Reid, J. K. (1970). A numerical method for locating the zeros and poles of a meromorphic function. In *Numerical Methods for Nonlinear Algebraic Equations* (Edited by P. Rabinowitz), Chap. 3, pp. 47–59. Gordon and Breach, London.

Anastasselou, E. G. and Ioakimidis, N. I. (1987). On the location of straight discontinuity intervals of arbitrary sectionally analytic functions by using complex path-independent integrals. *Comput. Meth. Appl. Mech. Engng* **65,** 165–176.

Brebbia, C. A., Telles, J. C. F. and Wrobel, L. C. (1984). *Boundary Element Techniques: Theory and Applications in Engineering*. Springer, Berlin.

Budiansky, B. and Rice, J. R. (1973). Conservation laws and energy-release rates. *J. Appl. Mech.* **40,** 201–203.

Chawla, M. M. and Ramakrishnan, T. R. (1974). Modified Gauss–Jacobi quadrature formulas for the numerical evaluation of Cauchy type singular integrals. *BIT* **14,** 14–21.

Churchill, R. V. and Brown, J. W. (1990). *Complex Variables and Applications* (5th Edn). McGraw-Hill, New York.

Copson, E. T. (1976). *An Introduction to the Theory of Functions of a Complex Variable* (13th impression). Oxford Univesity Press–Clarendon Press, Oxford.

Davis, P. J. and Rabinowitz, P. (1984). *Methods of Numerical Integration* (2nd Edn). Academic Press, Orlando, FL.

Dwight, H. B. (1961). *Tables of Integrals and Other Mathematical Data* (4th Edn). Macmillan, New York.

Erdogan, F. and Gupta, G. D. (1972). On the numerical solution of singular integral equations. *Q. Appl. Math.* **29,** 525–534.

Gradshteyn, I. S. and Ryzhik, I. M. (1980). *Table of Integrals, Series, and Products* (2nd English Edn). Academic Press, New York.

Hartmann, F. (1989). *Introduction to Boundary Elements: Theory and Applications*. Springer, Berlin.

Henrici, P. (1986). *Applied and Computational Complex Analysis, Volume Three: Discrete Fourier Analysis—Cauchy Integrals—Construction of Conformal Maps—Univalent Functions* (1st Edn). Wiley, New York.

Householder, A. S. (1970). *The Numerical Treatment of a Single Nonlinear Equation* (1st Edn). McGraw-Hill, New York.

Hromadka II, T. V. (1984). *The Complex Variable Boundary Element Method*. Springer, Berlin.

Hromadka II, T. V. and Pardoen, G. C. (1985). Application of the CVBEM to non-uniform St. Venant torsion. *Comput. Meth. Appl. Mech. Engng* **53,** 149–161.

Hsieh, C. K. and Kassab, A. J. (1991). Complex variable boundary element methods for the solution of potential problems in simply and multiply connected domains. *Comput. Meth. Appl. Mech. Engng* **86,** 189–213.

Ioakimidis, N. I. (1976). General methods for the solution of crack problems in the theory of plane elasticity. Doctoral Thesis at the National Technical University of Athens, Athens. [Available from University Microfilms, Ann Arbor, MI; Order No. 76-21,056; in Greek with an extensive English summary.]

Ioakimidis, N. I. (1980). On the evaluation of stress intensity factors in interface crack problems by using complex path-independent integrals. *Int. J. Fract.* **16,** R37–R41.

Ioakimidis, N. I. (1983). Application of the optical method of pseudocaustics to locating crack tips in plane elasticity problems. *Int. J. Fract.* **23,** R117–R120.

Ioakimidis, N. I. (1985a). Application of the generalized Siewert–Burniston method to locating zeros and poles of meromorphic functions. *Z. Angew. Math. Phys.* **36,** 733–742.

Ioakimidis, N. I. (1985b). Locating a straight crack in an infinite elastic medium by using complex path-independent integrals. *Acta Mech.* **57,** 241–246.

Ioakimidis, N. I. (1986a). Application of complex path-independent integrals to the solution of the problem of a straight crack in a finite plane isotropic elastic medium. *J. Elast.* **16,** 441–456.

Ioakimidis, N. I. (1986b). Determination of poles of sectionally meromorphic functions. *J. Comput. Appl. Math.* **15**, 323–327.

Ioakimidis, N. I. (1986c). A note on locating straight-crack tips in finite plane elastic media. *Int. J. Fract.* **32**, R11–R12.

Ioakimidis, N. I. (1987a). A new approach to the construction of some path-independent integrals about crack tips. In *Proceedings of the First National Congress on Mechanics*, Athens, 1986, Vol. I, pp. 459–469. Hellenic Society for Theoretical and Applied Mechanics and the Technical Chamber of Greece, Athens (in English).

Ioakimidis, N. I. (1987b). Quadrature methods for the determination of zeros of transcendental functions—a review. In *Numerical Integration: Recent Developments, Software and Applications* (Edited by P. Keast and G. Fairweather). Proceedings of a NATO Advanced Research Workshop, Halifax, Nova Scotia, 1986, pp. 61–82. D. Reidel, Dordrecht, The Netherlands.

Ioakimidis, N. I. (1987c). A class of surface-independent integrals in three-dimensional elasticity with an application to locating planar cracks. *Int. J. Fract.* **33**, R51–R54.

Ioakimidis, N. I. (1987d). Application of complex path-independent integrals to locating circular holes and inclusions in classical plane elasticity. Technical Report, School of Engineering, University of Patras.

Ioakimidis, N. I. (1988a). On the practical application of the method of complex path-independent integrals to problems of fracture mechanics. *Int. J. Fract.* **36**, R51–R54.

Ioakimidis, N. I. (1988b). Location of essential singularities of a class of analytic functions. *Int. J. Comput. Math.* **25**, 129–138.

Ioakimidis, N. I. (1990). Locating inclusions of the same material in finite plane isotropic elastic media by using complex path-independent integrals. *Strojn. Čas.* **41**, 353–364.

Ioakimidis, N. I. (1992a). Computation of the orders of singularity of sectionally analytic functions. *Appl. Math. Comput.* **48**, 13–19.

Ioakimidis, N. I. (1992b). Application of complex path-independent integrals to problems of bending of thin elastic plates. *Arch. Appl. Mech.* (*Ing.-Arch.*) **62**, 248–255.

Ioakimidis, N. I. (1992c). Locating branch points of sectionally analytic functions by using contour integrals and numerical integration rules. *Int. J. Comput. Math.* **41**, 215–222.

Ioakimidis, N. I. and Theocaris, P. S. (1977a). The problem of the simple smooth crack in an infinite anisotropic elastic medium. *Int. J. Solids Structures* **13**, 269–278.

Ioakimidis, N. I. and Theocaris, P. S. (1977b). Array of periodic curvilinear cracks in an infinite isotropic medium. *Acta Mech.* **28**, 239–254.

Ioakimidis, N. I. and Theocaris, P. S. (1978). Doubly-periodic array of cracks in an infinite isotropic medium. *J. Elast.* **8**, 157–169.

Ioakimidis, N. I. and Theocaris, P. S. (1979a). A system of curvilinear cracks in an isotropic elastic half-plane. *Int. J. Fract.* **15**, 299–309.

Ioakimidis, N. I. and Theocaris, P. S. (1979b). The second fundamental crack problem and the rigid line inclusion problem in plane elasticity. *Acta Mech.* **34**, 51–61.

Kassab, A. J. and Hsieh, C. K. (1990). Application of the complex variable boundary element method to solving potential problems in doubly connected domains. *Int. J. Numer. Meth. Engng* **29**, 161–179.

Muskhelishvili, N. I. (1963). *Some Basic Problems of the Mathematical Theory of Elasticity* (2nd English Edn). P. Noordhoff, Groningen, The Netherlands.

Nishimura, N. and Kobayashi, S. (1990). Regularized BIEs for crack shape determination problems. In *Boundary Elements XII* (Edited by M. Tanaka, C. A. Brebbia and T. Honma), Vol. 2, pp. 425–434. Computational Mechanics Publications, Southampton, and Springer, Berlin.

Nishimura, N. and Kobayashi, S. (1991). A boundary integral equation method for an inverse problem related to crack detection. *Int. J. Numer. Meth. Engng* **32**, 1371–1387.

Olver, P. J. (1984). Conservation laws in elasticity: II. Linear homogeneous isotropic elastostatics. *Arch. Rat. Mech. Anal.* **85**, 131–160.

Olver, P. J. (1985). Symmetry groups and path-independent integrals. In *Fundamentals of Deformation and Fracture* (Edited by B. A. Bilby, K. J. Miller and J. R. Willis). Eshelby Memorial Symposium, Sheffield, 2–5 April 1984. International Union of Theoretical and Applied Mechanics, pp. 57–71. Cambridge University Press, Cambridge.

Tanaka, M., Nakamura, M. and Nakano, T. (1990). Detection of cracks in structural components by the elastodynamic boundary element method. In *Boundary Elements XII* (Edited by M. Tanaka, C. A. Brebbia and T. Honma), Vol. 2, pp. 413–424. Computational Mechanics Publications, Southampton, and Springer, Berlin.

Theocaris, P. S. (1979). Experimental study of plane elastic contact problems by the pseudocaustics method. *J. Mech. Phys. Solids* **27**, 15–32.

Theocaris, P. S. and Ioakimidis, N. I. (1977a). A star-shaped array of curvilinear cracks in an infinite isotropic elastic medium. *J. Appl. Mech.* **44**, 619–624.

Theocaris, P. S. and Ioakimidis, N. I. (1977b). Numerical integration methods for the solution of singular integral equations. *Q. Appl. Math.* **35**, 173–183.

Theocaris, P. S. and Ioakimidis, N. I. (1979a). The problem of interaction between a misfitting inclusion and a crack in an infinite elastic medium. *J. Elast.* **9**, 97–103.

Theocaris, P. S. and Ioakimidis, N. I. (1979b). A remark on the numerical solution of singular integral equations and the determination of stress-intensity factors. *J. Engng Math.* **13**, 213–222.

Theocaris, P. S. and Ioakimidis, N. I. (1980a). A numerical method for the solution of plane crack problems in finite media. *Int. J. Math. and Math. Sci.* **3**, 739–760.

Theocaris, P. S. and Ioakimidis, N. I. (1980b). Stress-intensity factors and complex path-independent integrals. *J. Appl. Mech.* **47**, 342–346.

Theocaris, P. S., Ioakimidis, N. I. and Razem, C. I. (1981). The method of pseudocaustics for the experimental solution of simple elasticity problems. *Int. J. Mech. Sci.* **23**, 17–29.

Theocaris, P. S. and Razem, C. (1977). Deformed boundaries determined by the method of caustics. *J. Strain Anal.* **12**, 223–232.

Theocaris, P. S. and Tsamasphyros, G. (1980). On the complex path-independent integrals used in plane elasticity problems. *Proc. Acad. Athens* **55,** 441–472 (in English).

Tsamasphyros, G. (1981). On some path-independent integrals of plane elasticity. In *Mixed-Mode Crack Propagation* (Edited by G. C. Sih and P. S. Theocaris), pp. 269–281. Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands.

Tsamasphyros, G. (1989a). Mixed mode crack analysis using complex path-independent integrals. *Engng Fract. Mech.* **34,** 359–368.

Tsamasphyros, G. (1989b). Path-independent integrals in anisotropic media. *Int. J. Fract.* **40,** 203–219.

Tsamasphyros, G. J. and Theocaris, P. S. (1982a). A new concept of path-independent integrals for plane elasticity. *J. Elast.* **12,** 265–280.

Tsamasphyros, G. J. and Theocaris, P. S. (1982b). Path-independent integrals in inhomogeneous media. *Ing.-Arch.* **52,** 159–166.